We present here another method for solving the last question of exercise 3 in the Examen blanc:

We have a Markov chain $(X_n)_{n\geq 0}$ on $\{0,1,2,3\}$

$P = \begin{pmatrix} 0 & 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/2 & 0 & 1/4 \end{pmatrix}$. What is the expected number of times 2 is hit before the chain returns to 0 if $X_0 = 0$.

Solution: Let $A := \#\{n \geq 0 \mid X_n = 2,\ X_k \neq 0, k < n\}$. Using the Markov property, we have

$$\mathbb{E}[A \mid X_0 = i, X_1 = j] = \mathbb{E}[A \mid X_0 = i] \text{ for } i = 1,3,\ j = 1,2,3$$

For $i = 2$, we have by definition of A:

$$\mathbb{E}[A \mid X_0 = 2, X_1 = j] = 1 + \mathbb{E}[A \mid X_0 = j],\ j = 1,2,3.$$

For $j = 0$, we have

$$\mathbb{E}[A \mid X_0 = i, X_1 = 0] = \begin{cases} 0 & \text{if } i = 1,3 \\ 1 & i = 2. \end{cases}$$

Using this and the fact that $\mathbb{E}[A \mid X_0 = i] = \sum_j \mathbb{E}[A \mid X_0 = i, X_1 = j] p_{ij}$ and writing $l(i) := \mathbb{E}[A \mid X_0 = i]$, we get

$$\begin{aligned} l(0) &= \frac{1}{4} l(1) + \frac{1}{2} l(2) + \frac{1}{4} l(3) \qquad (1) \\ l(1) &= \frac{1}{4} l(2) + \frac{1}{4} l(3) \\ l(2) &= \frac{1}{4} + \frac{1}{4}(l(1)+1) + \frac{1}{4}(l(2)+1) + \frac{1}{4}(l(3)+1) \\ l(3) &= \frac{1}{2} l(1) + \frac{1}{4} l(3). \end{aligned}$$

Solving this, we get:

$$\begin{aligned} l(3) = \frac{2}{3} l(1),\ l(2) = \frac{10}{3} l(1), \\ \Longrightarrow \frac{10}{3} l(1) = 1 + \frac{1}{4} l(1) + \frac{5}{6} l(1) + \frac{1}{6} l(1) \\ \Longrightarrow l(1) = \frac{12}{25}, l(2) = \frac{8}{5}, l(3) = \frac{8}{25}. \end{aligned}$$

Replacing everything in (1), we finally get

$$l(0) = \frac{3}{25} + \frac{4}{5} + \frac{2}{25} = 1.$$